

## On evolutionary processes in large N quantum mechanics

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## On evolutionary processes in large N quantum mechanics

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ABSTRACT: Equations of motion of large N quantum mechanics are solved for infinite N in the case of unbroken global  $O(N)$  symmetry. It is shown that the only correction to the action is a change in the potential. All characteristics of the motion (frequencies of oscillations, times of rolling, a phase portrait etc.) can be computed by means of a classical mechanics for the corrected potential. In particular, a tunneling in the system is absent.

KEYWORDS: Field Theories in Lower Dimensions, Global Symmetries

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## 1 Introduction

Evolutionary processes have been studied in the framework of Quantum Mechanics from its early days. Even the first complete formulation of principles of Quantum Mechanics described a way to find a time dependence of observables (Heisenberg eqs). In Quantum Field Theory the common lore concerns mainly properties of vacua and perturbations around them. Still, time dependent processes are crucial in various branches of QFT (inflationary cosmology, QFT in curved spaces etc). These processes include a rolling of a quantum field from a maximum of a potential, oscillations around a minimum and a tunneling from one minimum of a potential to others.

In general, a study of time-dependent processes in QFT is prohibitively difficult. It can be carried out, however, in large  $N$  vector models.

A large  $N$  vector model with a global  $O(N)$  symmetry is a rare example of a field theory where an exact vacuum state can be found (for a review of large  $N$  models see [1]). This model has served as a toy model for various problems common in QFT. A three-dimensional scale-invariant model was shown to be quantumly conformal [2] and to possess a non-trivial phase structure [3].

General eqs. which govern time-dependent processes in the large  $N$  vector model were suggested in [4] and applied there to the case of  $d = 3$ . The applications included an exact rolling solution of a  $\phi^6$  model with vanishing energy and an approximate solutions for oscillations around minima in a general potential. In all cases the quantum processes were found to occur faster than their classical counterparts, signaling a presence of terms with time derivatives in addition to a known correction to a classical potential. A possible tunneling in the model was also studied and the tunneling amplitude was found to be larger than in the semiclassical computation.

The main difficulty in solving for time dependent processes in the large  $N$  model, as discussed in [4], is a necessity to find a Green's function of a differential operator which

itself contains an unknown function. In this paper we show that this problem can be solved in the case of  $d = 1$ , which corresponds to a large  $N$  quantum mechanics.

A large  $N$  quantum mechanics has been studied from various point of view. Numerous approximations have been used in order to derive systematically a  $1/N$  expansion (for a review and a comparison of some approximation technics see [5], in [6] yet another approach is suggested). A supersymmetric version of the large  $N$  quantum mechanics was also studied (for improvements that supersymmetry brings to the large  $N$  expansion see [7]). A certain supersymmetric version of the large  $N$  quantum mechanics was proposed as a description of a D-brane probe in the bubbling supertubes [8].

In this paper we solve EOM's derived in [4] for the quantum mechanics in the case of unbroken global  $O(N)$  symmetry. We show that the expectation value of the quantum field changes in time according to a classical EOM with a modified potential, which means that the effective action in this case does not involve any corrections to the kinetic term, the only correction being therefore that to the potential. All characteristics of the motion (like frequencies of oscillations around minima of the effective potential and rolling times in that potential, and possible types of the motion in general) can be derived by means of the classical mechanics. In addition, a tunneling is completely suppressed.

The paper is organized as follows. In section 2 we review the large  $N$  models in general and introduce eqs. that govern a time evolution of the quantum system. In section 3 we rewrite a classical EOM in a form which is convenient for a comparison with the quantum case. In section 4 we solve the quantum EOM's. Section 5 is a summary of the results. We end up with an appendix A where we show that there is no tunneling in the system in the limit  $N \rightarrow \infty$  and confirm this result by conventional methods of quantum mechanics.

## 2 Scalar model in the large N limit. A review

Let us consider an  $O(N)$ -symmetric Euclidean action for an  $N$ -component scalar field  $\vec{\phi}$  in  $d$  space-time dimensions

$$S(\vec{\phi}) = \int \left[ \frac{1}{2} (\partial_\mu \vec{\phi})^2 + NU \left( \frac{\vec{\phi}^2}{N} \right) \right] d^d x. \tag{2.1}$$

The potential has a Taylor expansion of the form

$$U \left( \frac{\vec{\phi}^2}{N} \right) = \sum_{n=1}^{\infty} \frac{g_{2n}}{2n} \left( \frac{\vec{\phi}^2}{N} \right)^n, \tag{2.2}$$

with  $g_{2n}$  kept fixed as  $N \rightarrow \infty$ . The corresponding partition function is

$$Z = \int D\vec{\phi} e^{-S(\vec{\phi})}. \tag{2.3}$$

In order to use the fact that  $\vec{\phi}$  has many components insert the following representation of unity into  $Z$ :

$$1 \sim \int D\rho \delta(\vec{\phi}^2 - N\rho) \sim \int D\rho D\lambda e^{-i \int \frac{\lambda}{2} (\vec{\phi}^2 - N\rho) d^d x}. \tag{2.4}$$

Now one can integrate over  $\vec{\phi}$  and obtain

$$Z = \int D\rho D\lambda e^{-NS_{\text{eff}}(\rho, \lambda)}, \quad (2.5)$$

where

$$S_{\text{eff}}(\rho, \lambda) = \frac{1}{2} \int [2U(\rho) - i\lambda\rho] d^d x + \frac{1}{2} \text{Tr} \ln(-\square + i\lambda). \quad (2.6)$$

When  $N$  is large, this form of the path integral suggests to use the saddle point method to calculate the integral. The two saddle point equations obtained by varying the auxiliary fields  $\rho$  and  $\lambda$  are<sup>1</sup>

$$2U'(\rho) = i\lambda, \quad \rho = \text{tr} \frac{1}{-\square + i\lambda}, \quad (2.7)$$

This form of the saddle point equations is convenient in the case of constant fields. These constant solutions can be found as extrema of the effective potential

$$U_{\text{eff}}(\rho) = U(\rho) + \frac{2-d}{2d} \Gamma\left(1 - \frac{d}{2}\right)^{-\frac{2}{d-2}} (4\pi\rho)^{\frac{d}{d-2}}, \quad (2.8)$$

together with the following value of the field  $\lambda$ :

$$i\lambda = \left[ \frac{(4\pi)^{d/2} \rho}{\Gamma\left(1 - \frac{d}{2}\right)} \right]^{\frac{2}{d-2}}. \quad (2.9)$$

The necessity of the second term in (2.8) is most clearly seen in  $d = 1$ , where the theory becomes a quantum mechanics. In this case the effective potential is

$$U_{\text{eff}}(\rho) \Big|_{d=1} = U(\rho) + \frac{1}{8\rho}. \quad (2.10)$$

Then, consider the case of  $U = \frac{g_2}{2}\rho$ , which corresponds to a system of  $N$  decoupled harmonic oscillators. The last term in (2.8) shifts a minimum of the potential from  $\rho = 0$  to a correct value  $\rho = \frac{1}{2\sqrt{g_2}}$ . Therefore this term takes into account a spreading of the ground state wave function.

If the fields are not constant then, as shown in [4], the correct form of the equations is

$$2U'(\rho(x)) = i\lambda(x), \quad \rho(x) = G(x, x), \quad \left(-\square_x + i\lambda(x)\right) G(x, y) = \delta(x - y) \quad (2.11)$$

where  $\square_x$  is a Laplacian w.r.t.  $x$ .

A major role in the above equations is played by the Green's function  $G(x, y)$ . The field  $\rho$  is equal to this function at coincident points. In QFT this is divergent and is to be regularized. We will concentrate on the case of  $d = 1$  where no regularization is needed.

The main difficulty in solving equations (2.11) stems from the fact that  $G(x, y)$  is a Green's function of the operator which involves an unknown function  $\lambda(x)$ . In [4] there was found a particular exact solution of these equations for  $d = 3$  and a potential  $U(\rho) = \frac{g_6}{6}\rho^3$ , which corresponds to a model  $\vec{\phi}^6$ . The particular solution found there corresponds to a

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<sup>1</sup>We use here the definition  $\text{Tr} = \int d^D x \text{tr}$

vanishing energy. In that case there is no scale in the problem since the coupling constant  $g_6$  is dimensionless in  $d = 3$ . Therefore the form of the solution is determined by dimensional analysis up to dimensionless multiplicative constants which can be fixed.<sup>2</sup> In this paper we solve the equations (2.11) in the case of  $d = 1$ .

### 3 Classical analysis

In this section we consider a classical version of the large  $N$  quantum mechanics. Our main purpose is to write the EOM in a form that will be convenient for a comparison with its quantum counterpart. For the same reason we work in the Euclidean signature. We denote the Euclidean time by  $\tau$ .

The Euclidean Lagrangian of the system is

$$L = \frac{1}{2} \left( \partial_\tau \vec{\phi} \right)^2 + NU \left( \frac{\vec{\phi}^2}{N} \right). \quad (3.1)$$

We consider a phase with an unbroken global  $O(N)$  symmetry, which means that the field configurations we are interested in are of the form

$$\vec{\phi}(\tau) = \phi(\tau) (1, 1, \dots, 1), \quad (3.2)$$

where the vector on the r.h.s. has  $N$  components. For such field configurations the Lagrangian can be rewritten as

$$L = N \left( \frac{1}{2} \dot{\phi}^2 + U(\phi^2) \right), \quad (3.3)$$

where a dot is a derivative w.r.t.  $\tau$ . The Euclidean EOM is

$$\ddot{\phi} = \frac{dU}{d\phi}. \quad (3.4)$$

In order to make this equation resemble the quantum one we introduce a classical analog of the field  $\rho$ :

$$\rho_{cl} = \frac{\vec{\phi}^2}{N} = \phi^2. \quad (3.5)$$

Then, using the fact that  $\frac{dU}{d\phi} = 2\phi \frac{dU}{d\rho}$  one can write a differential equation of the third order for  $\rho_{cl}$ :

$$\ddot{\rho}_{cl} = 8U'(\rho_{cl}) \dot{\rho}_{cl} + 4U''(\rho_{cl}) \rho_{cl} \dot{\rho}_{cl}. \quad (3.6)$$

This equation should be supplemented by initial conditions. Suppose that at  $\tau = 0$  the field  $\phi$  is equal to  $\phi_0$  and its time derivative is  $\dot{\phi}_0$ . Then the initial conditions for eq. (3.6) are

$$\rho_{cl}(0) = \phi_0^2, \quad \dot{\rho}_{cl}(0) = 2\phi_0 \dot{\phi}_0, \quad \ddot{\rho}_{cl}(0) = U'(\phi_0^2) + 2\dot{\phi}_0^2. \quad (3.7)$$

This is the EOM in the form of (3.6) that will be compared to the quantum one in the next section.

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<sup>2</sup>In [4] there were also considered approximate solutions of equations (2.11)

## 4 Quantum solution

In this section we derive the quantum EOM for the field  $\rho$  starting from general equations (2.11). The main result is that the quantum EOM is again of the form (3.6), but the initial conditions are different. It will also be shown that a difference in the initial conditions can be turned into a correction to a potential.

In the case of quantum mechanics eqs. (2.11) reduce to

$$2U'(\rho(\tau)) = i\lambda(\tau), \quad \rho(\tau) = G(\tau, \tau), \quad \left(-\partial_\tau^2 + i\lambda(\tau)\right) G(\tau, \tau_0) = \delta(\tau - \tau_0), \quad (4.1)$$

and there is no need in a regularization.

We are going to consider a rolling of the system from a top of the potential. Therefore we assume that the fields  $\rho$  and  $\lambda$  possess limits when  $\tau \rightarrow \infty$ . We will denote the limit of  $i\lambda$  by  $m^2$  and assume in what follows that its value is positive. Then we redefine the field  $\lambda$ :

$$i\lambda = i\lambda_1 + m^2, \quad (4.2)$$

so that the new field  $\lambda_1$  goes to 0 at infinity. In terms of this new field the eqs. (4.1) become

$$2U'(\rho(\tau)) = i\lambda_1(\tau) + m^2, \quad (4.3)$$

$$\rho(\tau) = G_m(\tau, \tau), \quad (4.4)$$

$$\left(-\partial_\tau^2 + i\lambda_1(\tau) + m^2\right) G_m(\tau, \tau_0) = \delta(\tau - \tau_0), \quad (4.5)$$

where the subscript of  $G$  indicates the value of the parameter  $m$  at which the Green's function is computed.

In one dimension the Green's function is closely related to solutions of the corresponding homogeneous equation

$$\left(-\partial_\tau^2 + i\lambda_1(\tau) + m^2\right) g(\tau) = 0. \quad (4.6)$$

Since  $\lambda_1$  vanishes at infinity the solutions of this equation look at infinity as  $e^{\pm m\tau}$  (this approach is similar to that used in [11]). We choose two linear independent solutions  $g_{1,2}$  of the homogeneous equation so that the function  $g_1(\tau)$  goes to 0 as  $\tau \rightarrow -\infty$  and  $g_2(\tau)$  goes to 0 as  $\tau \rightarrow +\infty$ . We will fix their normalization later.

In terms of these solutions the Green's function can be written as

$$G_m(\tau, \tau_0) = \begin{cases} \alpha_1 g_1(\tau), & \tau < \tau_0 \\ \alpha_2 g_2(\tau), & \tau > \tau_0 \end{cases} \quad (4.7)$$

where the coefficients  $\alpha_{1,2}$  satisfy

$$\alpha_1 g_1(\tau_0) = \alpha_2 g_2(\tau_0), \quad \alpha_1 g_1'(\tau_0) - \alpha_2 g_2'(\tau_0) = 1. \quad (4.8)$$

Solving these equations leads to the following expression for the Green's function:

$$G_m(\tau, \tau_0) = \frac{1}{W(g_1, g_2)} \begin{cases} g_1(\tau) g_2(\tau_0), & \tau < \tau_0 \\ g_1(\tau_0) g_1(\tau), & \tau > \tau_0 \end{cases} \quad (4.9)$$

where  $W(g_1, g_2) \equiv g_1'(\tau)g_2(\tau) - g_2'(\tau)g_1(\tau)$  is a Wronskian of the two solutions. It is independent of  $\tau$  and we fix the normalization of the basic solutions  $g_{1,2}(\tau)$  so that  $W(g_1, g_2) = 1$ . With this normalization

$$G_m(\tau, \tau_0) = \begin{cases} g_1(\tau) g_2(\tau_0), & \tau < \tau_0 \\ g_1(\tau_0) g_1(\tau), & \tau > \tau_0 \end{cases} \quad (4.10)$$

and the Green's function with coincident points (which is equal to  $\rho$ ) is

$$G_m(\tau, \tau) \equiv \rho(\tau) = g_1(\tau)g_2(\tau). \quad (4.11)$$

Using this form of  $\rho$  and eq. (4.6) we write the differential equation for  $\rho$ :

$$\ddot{\rho} = 4(i\lambda_1 + m^2) \dot{\rho} + 2i\dot{\lambda} \rho. \quad (4.12)$$

Now, using (4.3) we get

$$\ddot{\rho} = 8U'(\rho) \dot{\rho} + 4U''(\rho) \rho \dot{\rho}. \quad (4.13)$$

This equation precisely coincides with the classical equation (3.6). However, this does not mean that the possible motions of the system are the same, since we need to specify initial conditions. Eqs. (3.6), (4.13) are of the third order and should be integrated once to be brought to a usual form of equations of dynamics.

In order to integrate once the eq. (4.13) define a new field in analogy with the classical case:

$$\Phi = \sqrt{\rho}. \quad (4.14)$$

In terms of this new field a first integral of eq. (4.13) can be written as

$$\ddot{\Phi} = \frac{dU}{d\Phi} + \frac{\alpha}{\Phi^3}. \quad (4.15)$$

Here  $\alpha$  is an integration constant. In order to fix it consider constant solutions of (4.15). They are given by extrema of the function

$$\bar{U}(\Phi^2) = U(\Phi^2) - \frac{\alpha}{2\Phi^2}, \quad (4.16)$$

or if we rewrite it in terms of  $\rho$ , by extrema of

$$\bar{U}(\rho) = U(\rho) - \frac{\alpha}{2\rho}. \quad (4.17)$$

We see that the function  $\bar{U}$  is of the form of a general effective potential of the large  $N$  vector model at  $d = 1$ , which is given in (2.10). This fixes  $\alpha = -1/4$ , and the effective potential becomes

$$U_{\text{eff}}(\Phi^2) = U(\Phi^2) + \frac{1}{8\Phi^2}. \quad (4.18)$$

The quantum EOM can be written in terms of the field  $\Phi$  as

$$\ddot{\Phi} = \frac{dU(\Phi^2)}{d\Phi} - \frac{1}{4\Phi^3}, \quad (4.19)$$



or, Wick rotating back to the Lorentzian time  $t$ ,

$$\ddot{\Phi} = -\frac{dU(\Phi^2)}{d\Phi} + \frac{1}{4\Phi^3}, \tag{4.20}$$

This equation by construction correctly reproduces the extrema on the quantum effective potential (2.10). Other solutions of this equation describe time-dependent processes in the system.

We have proven that the quantum mechanical EOM for an expectation value of the field  $\Phi$  (or  $\rho$ ) is of a form of the classical EOM but with a corrected potential. Therefore a classical intuition about a motion of the system applies directly to the quantum case. For example, a frequency of small oscillations around a minimum of the effective potential is given by  $\omega^2 = U''_{\text{eff}}(\Phi_0)$ , where  $\Phi_0$  is an expectation value of the field  $\sqrt{\rho}$  at the minimum of  $U_{\text{eff}}$ . One can also draw a phase portrait on the phase plane which would describe qualitatively all possible motions of the system. In addition, the quantum mechanical tunneling is not allowed. This is a consequence of the large  $N$  limit. We show this in a different way in appendix A, where we also show that the tunneling amplitude goes to zero as  $\pi^{N/2} [(N/2)!]^{-1}$ .

## 5 Summary and discussion

In this paper we considered the large  $N$  quantum mechanics with an unbroken global  $O(N)$  symmetry. We found that the mean value of the square of the field satisfies a classical EOM with modified potential (4.18). Since the EOM is purely classical we conclude that the correction to the potential is the only difference between the classical action and the 1PI effective action which governs the quantum evolution. In particular, there are no terms with time derivatives, contrary to the case of  $d = 3$ , where such terms are present [4]. This result allows one to compute all characteristics of time-dependent processes in an arbitrary potential (frequencies of oscillations around minima of the potential, characteristic times of rolling, the phase portrait etc) by the conventional means of the classical mechanics. It also follows that the tunneling in the system is suppressed in the large  $N$  limit.

A desirable continuation of this work is to find a way to solve the quantum EOM's in the case of higher dimensionality. This is much more difficult, however. Although a reduction of the problem to a one-dimensional case is always possible (assuming that the solution we are looking for depends only on time), an expression for the field  $\rho$  in terms of solutions of a homogeneous eq. similar to (4.6) will in general involve an integral over a mass, a fact that significantly complicates a computation.

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## A Absence of tunneling

In this appendix we use our results in order to show that there is no tunneling in the system if the global  $O(N)$  symmetry is unbroken.

As shown in section 4, the dependence on time of the expectation value of  $\phi^2$  is governed by the classical EOM with the effective potential (4.20). Therefore, if the field is at a minimum of the effective potential the value of  $\langle \phi^2 \rangle$  cannot change, and the tunneling is impossible. In order to see this in a different way we change our point of view and interpret our problem as a problem of a motion of a particle of mass 1 in an  $N$ -dimensional space with a spherically-symmetric potential  $NU(r^2/N)$ . Components of the field  $\vec{\phi}$  are interpreted as coordinates of the particle. We consider a situation when the global  $O(N)$  symmetry is unbroken, which means that the wave function of the particle is spherically symmetric (s-wave). The radial Schrodinger eq. is

$$-\frac{1}{2} \frac{1}{r^{N-1}} \frac{d}{dr} r^{N-1} \frac{d}{dr} \Psi(r) + NU \left( \frac{r^2}{N} \right) \Psi(r) = E \Psi(r). \quad (\text{A.1})$$

Now make the following redefinitions: Define a radial wave function as

$$f(r) = r^{\frac{N-1}{2}} \Psi(r), \quad (\text{A.2})$$

introduce a new variable  $\xi = \frac{r}{\sqrt{N}}$  and assume that  $N \gg 1$ . The eq. (A.1) becomes

$$-\frac{1}{2N^2} \frac{d^2}{d\xi^2} f(r) + \left[ \frac{1}{8\xi^2} + U(\xi^2) \right] f(\xi) = \frac{E}{N} f(\xi). \quad (\text{A.3})$$

This eq. describes a motion of a particle of a large mass  $N^2$  in the effective potential (4.18). All eigenvalues go to 0 as  $N \rightarrow \infty$ , hence the solutions of this eq. have a small spread and therefore a tunneling is impossible.

Yet another way to see this is to carry out the instanton computation. The amplitude of tunneling is

$$\Gamma = A e^{-B}, \quad (\text{A.4})$$

where  $B$  is the action of a instanton and the coefficient  $A$  is proportional to a volume of the symmetry manifold [9, 10]. In our case this manifold is an  $N - 1$ -dimensional sphere. Its volume is  $V = 2\pi^{\frac{N-1}{2}} / \Gamma(\frac{N-1}{2})$ , and as  $N \rightarrow \infty$  the volume  $V \rightarrow 0$ . So the tunneling amplitude vanishes in this limit.

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